Quaternion Grassmann-Hamilton-Clifford algebras: new mathematical tools for classical and relativistic modeling

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Abstract— **The paper presents new mathematical tools for classical and relativistic modeling based on a quaternion formulation of Clifford algebras which we shall call Grassmann-Hamilton-Clifford algebras. These algebras allow to develop an associative exterior calculus for any metric and any dimension yielding probably the best representations of the covariance groups. As applications, the paper develops these algebras in euclidean three-space and pseudo-euclidean space-time, and in particular the Frenet frame and the relativistic moving frame.**

Keywords— **Quaternions, biquaternions, tetraquaternions, Grassmann-Hamilton-Clifford algebras, classical and relativistic modeling.**

I. INTRODUCTION

In 1843, William Rowan Hamilton (1805-1865) discovered the quaternions and, later on, was to introduce biquaternions, defined below. About the same time, Hermann Günther Grassmann (1809-1877), whose 200th anniversary of his birth is celebrated this year, presented in his Ausdehnungslehre (1844) a new n-dimensional associative multivector calculus. William Kingdom Clifford (1845-1879) was to demonstrate in 1878 that any Clifford algebra, is directly linked to quaternions. Though the use of Clifford algebras in physics and engineering has grown rapidly in recent years, most developments have priviledged a geometric approach, whereas the author uses an algebraic approach based on quaternions. The aim of this paper is to present the Grassmann-Hamilton-Clifford algebras in 1, 2, 3 and four dimensions and to show their potential for classical and relativistic modeling.

II. QUATERNION GRASSMANN-HAMILTON-CLIFFORD ALGEBRAS

1. Quaternions, biquaternions and tetraquaternions

a. Quaternions $\boldsymbol{H}$ (1843)

A quaternion is a set of four real numbers

$$a = a_0 + a_1 i + a_2 j + a_3 k = \left(a_0, a_1, a_2, a_3\right) = \left(a_0, \vec{a}\right)$$

where i, j, k multiply according to the rules

$$i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, etc.$$

The product of two quaternions a and b is defined by

$$ab = \left(a_0 b_0 - \vec{a}.\vec{b}, a_0\vec{b} + b_0\vec{a} + \vec{a}\times\vec{b}\right)$$

with the usual expressions of the scalar and vector products

$$\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec{a}\times\vec{b} = \left(a_2 b_3 - a_3 b_2\right)i + \left(a_3 b_1 - a_1 b_3\right)j + \left(a_1 b_2 - a_2 b_1\right)k.$$

R, **C**, **H** are fields without zero divisors (i.e., $ab = 0 \Rightarrow$ a or b=0); there exist no others. The classical vector calculus was obtained at the end of the nineteenth century from the quaternion product by taking $a_0 = b_0 = 0$ and separating the dot and vector products.

b. Biquaternions $\boldsymbol{H} \otimes \boldsymbol{C}$

A biquaternion is a set of two quaternions

$$\alpha = \alpha_0 + \alpha_1 I, \alpha_i \in \boldsymbol{H}$$

where I commutes with the small i, j, k of the α_i ($I^2 = -1$).

c. Tetraquaternions $\boldsymbol{H} \otimes \boldsymbol{H}$

A tetraquaternion is a quaternion having quaternions as coefficients

$$\alpha = \alpha_0 + \alpha_1 I + \alpha_2 J + \alpha_3 K, \alpha_i \in \boldsymbol{H}$$

where the small i, j, k of the α_i commute with the capital I, J, K.

2. Grassmann-Clifford exterior algebra

a. Definition

Grassmann had the fundamental idea of an algebra composed of n generators $e_1, e_2, ..., e_n$ multiplying according to the rule $e_i e_j = -e_j e_i \left(i \neq j\right)$; Clifford was to add the requirement $e_i^2 = \pm 1$ which completed the precise definition of the Grassmann-Clifford algebra C_n. The 2^n ele-

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ments of the algebra are composed of the n generators, the various products $e_ie_j, e_ie_je_k, \ldots$ and the unit element 1. C^+ is the subalgebra constituted by products of an even number of e_i.

b. Interior and exterior products

Once the product of two general elements of the algebra is defined, one can define interior and exterior products of two vectors $a = a_1e_1 + a_2e_2 + \ldots + a_ne_n$ (similarly for b) via the formulas

$$ab = \lambda(a.b) + \mu(a \wedge b), ba = \lambda(a.b) - \mu(a \wedge b)$$

where $\lambda = \mu = \pm 1$ are given coefficients and with the assumption that, by definition, $a.b = b.a, a \wedge b = -b \wedge a$, one thus obtains

$$2a.b = \lambda^{-1}(ab + ba), 2a \wedge b = \lambda^{-1}(ab - ba).$$

A multivector (bivector, trivector, etc.)

$$A_p = a_1 \wedge a_2 \wedge \ldots \wedge a_p \left(2 \leq p \leq n\right)$$

where a_p are vectors, is then defined by recurrence

$$2a.A_p = \lambda^{-p}\left[aA_p - (-1)^p A_pa\right]$$

$$2a \wedge A_p = \lambda^{-p}\left[aA_p + (-1)^p A_pa\right];$$

by definition, one takes

$$A_p.a = (-1)^{p-1} a.A_p, A_p \wedge a = (-1)^p a \wedge A_p.$$

An important property of the exterior product is its associativity $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ where a, b, c are vectors.

Other interior and exterior products between multivectors are defined by

$$A_p \wedge B_q = a_1 \wedge \left(a_2 \wedge \ldots \wedge a_p \wedge B_q\right);$$

and for $p \leq q$,

$$A_p.B_q = \left(a_1 \wedge \ldots a_{p-1}\right).\left[a_p.B_q\right];$$

by definition, one assumes $A_p.B_q = (-1)^{p(q+1)} B_q.A_p$.

The above formulas allow to define the various interior and exterior products in a Grassmann-Clifford algebra whatever the dimension of the space.

3. Clifford theorem (1878)

a. Theorem

If n=2m (m integer), the Grassmann-Clifford algebra C_{2m} is the tensor product of m quaternion algebras. If n=2m-1, the Grassmann-Clifford algebra is the tensor product of m-1 quaternion algebras and the algebra $(1, \omega)$ where ω is the product of the 2m generators $(\omega = e_1e_2 \ldots e_{2m})$ of the algebra C_{2m}.

b. Examples

Table 1: Examples

	Grahamclif algebras	Generators
n=1	$C_1 = \boldsymbol{C}$ complex numbers	$e_1 = i$ $e_1^2 = -1$
n=2	$C_2 = \boldsymbol{H}$ quaternions	$e_1 = j, e_2 = k$ $e_1^2 = e_2^2 = -1$
n=3	$C_3 = \boldsymbol{H} \otimes \boldsymbol{C}$ biquaternions	$e_1 = iI, e_2 = jI, e_3 = kI$ $(I^2 = -1)$ $e_1^2 = e_2^2 = e_3^2 = 1$
n=4	$C_4 = \boldsymbol{H} \otimes \boldsymbol{H}$ tetraquaternions	$e_0 = j \otimes 1 = j,$ $e_1 = k \otimes i = kI, (I = 1 \otimes i)$ $e_2 = k \otimes j = kJ,$ $e_3 = k \otimes k = kK$ $e_0^2 = -1, e_1^2 = e_2^2 = e_3^2 = 1$

Table 1 above proves the Clifford theorem for n=1, 2, 3, 4. Any Grassmann-Clifford algebra being related to quaternions, it seems to us appropriate to call them Grassmann-Hamilton-Clifford algebras (or Grahamclif algebras, for short). Furthermore, Table 1 shows that quaternions allow to construct an exterior calculus only for a two-dimensional space. Since the classical vector calculus was derived form quaternions at the end of the nineteenth century, one might perhaps realize that the former is not necessarily the most coherent calculus for 3-D classical physics. We shall present below an exterior calculus in 3-D classical physics and then a relativistic one in 4-D.

III. EUCLIDEAN 3-SPACE

The metric is $ds^2 = dx^2 + dy^2 + dz^2$

1. Grahamclif algebra: $\boldsymbol{H} \otimes \boldsymbol{C}$ (biquaternions)

The algebra contains 8 elements: 3 for the vectors, 3 for bivectors (surfaces, torques, etc.), 1 for volumes and 1 for scalars.

1	$i = e_3 e_2$	$j = e_1 e_3$	$k = e_2 e_1$
$I = e_1 e_2 e_3$	$iI = e_1$	$jI = e_2$	$kI = e_3$

with $e_1^2 = e_2^2 = e_3^2 = 1$, $e_1 e_2 = -e_2 e_1, etc.$ and $I^2 = -1$. The general element A of the algebra is

$$A = (a_0 + a_1 i + a_2 j + a_3 k) + I(b_0 + b_1 i + b_2 j + b_3 k);$$

the conjugate of A is given by

$$A_c = (a_0 - a_1 i - a_2 j - a_3 k) + I(b_0 - b_1 i - b_2 j - b_3 k)$$

with $(AB)_c = (B)_c (A)_c$. The dual of A is $A^* = IA$.

2. Vectors

A vector is given by $x = x_1 e_1 + x_2 e_2 + x_3 e_3$ with $x^2 = x_1^2 + x_2^2 + x_3^2$ yielding a scalar. A unit vector satisfies $x^2 = 1$.

3. Interior and exterior products

Given two vectors x, y one defines

$$x.y = \frac{xy + yx}{2} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$x \wedge y = \frac{xy - yx}{2} = (x_3 y_2 - x_2 y_3) i + (-x_3 y_1 + x_1 y_3) j + (x_2 y_1 - x_1 y_2) k$$

The other products are defined according to the rules given in II. 2) with $\lambda = \mu = 1$.

4. Orthogonal projection of a vector u on a vector a

Writing $u = u_p + u_o$ (where p stands for parallel and o for orthogonal) one has

$$u_p = (u.a) a^{-1}, u_o = (u \wedge a) a^{-1}$$

5. Orthogonal symmetry with respect to a hyperplane

By definition, a hyperplane is a subspace of dimension (n-1). Taking a unit vector a ($a^2 = 1$) perpendicular to the hyperplane (a plane in this case) going through the origin, one has for the symmetrical vector x' of x, $x' = -axa$

6. Rotation

Combining the orthogonal symmetries, one obtains for a conical rotation of the vector x by an angle θ around the unit vector $I\vec{u}$ ($\vec{u} = u_1 i + u_2 j + u_3 k$, $u_1^2 + u_2^2 + u_3^2 = 1$) the formula $x' = r x r_c$ with $r = \left(\cos\frac{\theta}{2} + \vec{u}\sin\frac{\theta}{2}\right) \in C^+$ ($r r_c = 1$).

One verifies the conservation of the norm $x' x'_c = x x_c$. The same formula holds for any element A of the algebra $A' = r A r_c$.

7. Frenet frame

Consider the vector $OM = x(t) e_1 + y(t) e_2 + z(t) e_3$ and $ds^2 = dx^2 + dy^2 + dz^2$, $u = \frac{dOM}{ds} = e_{1m}$. One has the Frenet relations

$$\frac{de_{1m}}{ds} = \frac{e_{2m}}{R}, \frac{de_{2m}}{ds} = -\frac{e_{1m}}{R} + \frac{e_{3m}}{T}, \frac{de_{3m}}{ds} = -\frac{e_{2m}}{T}$$

and $B_1 = u \wedge \frac{du}{ds}$, $B_1 B_{1c} = \frac{1}{R^2}$,

$T_1 = u \wedge \frac{du}{ds} \wedge \frac{d^2 u}{ds^2}$, $T_1 T_{1c} = -\frac{1}{R^4 T^2}$ yielding immediately the invariants R and T. Biquaternions thus allow a Grahamclif algebra modeling of the euclidean space in 3 dimensions within classical physics.

IV. PSEUDO-EUCLIDEAN 4-SPACE

Since Einstein (1905), the metric of relativistic physics is no longer 3D-euclidean but 4-D pseudo-euclidean. The metric is $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ necessitating the use of another Grahmclifalgebra.

1. Grahamclif algebra $\boldsymbol{H} \otimes \boldsymbol{H}$ (tetraquaternions)

This algebra having been presented extensively together with physical applications in the books by Girard [1-2], we shall only give a few elements. In four dimensions, the algebra contains $2^4 = 16$ elements: 4 elements for the vectors, 6 for the bivectors (surfaces, torques, electromagnetic fields, etc.), 4 for trivectors (trivolumes, etc.), 1 for pseudo-scalars and 1 for scalars.

1	$I = e_3 e_2$	$J = e_1 e_3$	$K = e_2 e_1$
i	$iI = e_0 e_1$	$iJ = e_0 e_2$	$iK = e_0 e_3$
$j = e_0$	$jI = e_0 e_3 e_2$	$jJ = e_0 e_1 e_3$	$jK = e_0 e_2 e_1$
$k = e_1 e_2 e_3$	$kI = e_1$	$kJ = e_2$	$kK = e_3$

with $e_0^2 = -1, e_1^2 = e_2^2 = e_3^2 = 1$, $e_1 e_2 = -e_2 e_1, etc.$

The general element of the algebra A is

$$A = \left(a + ib + jc + kd; \vec{m} + i\vec{n} + j\vec{r} + k\vec{s}\right)$$

wth $\vec{m} = m_1 I + m_2 J + m_3 K, etc.$; the conjugate of A is

$$A_c = \left(a + ib - jc + kd; -\vec{m} - i\vec{n} + j\vec{r} - k\vec{s}\right)$$

With $\left(AB\right)_c = \left(B_c\right)\left(A_c\right)$. The dual of A is $A^* = iA$.

2. Vectors

A vector is of the type $x = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3$ where $x^0 = ct$ and $xx_c = \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2$. A unit time like vector u satisfies $uu_c = 1$ and a unit space like vector $uu_c = -1$.

3. Interior and exterior products

Taking two vectors x, y one defines

$$x.y = -\frac{xy + yx}{2} = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

$$x \wedge y = -\frac{xy - yx}{2}.$$

The other products are defined according to the formulas of II. 2) with $\lambda = \mu = -1$.

4. Lorentz group

From the orthogonal symmetries with respect to hyperplanes, one obtains for the orthochronous Lorentz group with x', x being vectors, $x' = axa_c$, with

$aa_c = 1, a \in C^+, (a = br$ or $rb)$,

$$r = \left(\cos\frac{\theta}{2} + \vec{u}\sin\frac{\theta}{2}\right) \in C^+$$

$$b = \left(Cosh\frac{\varphi}{2} + i\vec{v}Sinh\frac{\varphi}{2}\right) \in C^+$$

$\vec{u} = u^1 I + u^2 J + u^3 K$, $\left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2 = 1$ and similarly for $\vec{v}$. The same formula holds for any element A of the algebra $A' = aAa_c$.

5. Relativistic moving frame

Let $OM = ct\left(\tau\right) + x(\tau)e_1 + y(\tau)e_2 + z(\tau)e_3$ be a vector with $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ and $u = \frac{dOM}{ds} = e_{0m}$. The relativistic moving frame is defined by the relations

$$\frac{de_{0m}}{ds} = \frac{e_{1m}}{R_1}, \frac{de_{1m}}{ds} = \frac{e_{0m}}{R_1} + \frac{e_{2m}}{R_2},$$

$$\frac{de_{2m}}{ds} = -\frac{e_{1m}}{R_2} + \frac{e_{3m}}{T}, \frac{de_{3m}}{ds} = -\frac{e_{2m}}{T}$$

where $1/R_1 \rangle 0, 1/R_2 \rangle 0$ are respectively the first and second curvatures and $1/T$ the torsion. One has

$$B_1 = u \wedge \frac{du}{ds},\ B_1 B_{1c} = -\frac{1}{R_1^2},$$

$$T_1 = u \wedge \frac{du}{ds} \wedge \frac{d^2u}{ds^2},\ T_1 T_{1c} = \frac{1}{R_1^4 R_2^2},$$

$$P_1 = u \wedge \frac{du}{ds} \wedge \frac{d^2u}{ds^2} \wedge \frac{d^3u}{ds^3} = \frac{i}{R_1^3 R_2^2 T}$$

$P_1 P_{1c} = -\frac{1}{R_1^6 R_2^4 T^2}$; these relations yield immediately the relativistic invariants R_1, R_2, T. Tetraquaternions thus allow a Grahamclif algebra modeling of relativistic physics.

V. CONCLUSION

The paper has presented the Grassmann-Hamilton-Clifford algebras in general and applications in 3-D and 4-D in particular. We hope to have shown that these algebras constitute coherent mathematical tools for classical and relativistic modeling.

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